

Boundary layer at a swept stagnation line of semi-infinite extent

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A three-dimensional boundary layer developing along a semi-infinite swept stagnation line from a starting edge and evolving into that associated with such a line of infinite extent is calculated. A series solution useful for assessing the counteracting effects of cross-flow and mass transfer near the starting edge and for providing initial data for a subsequent streamwise, numerical solution is developed. The asymptotic behaviour far from the starting edge is examined and shown to involve only eigenfunction contributions associated with the far upstream flow. However, it is not presently possible to determine the relevant eigenvalues and eigenfunctions. Numerical solutions based on a difference-differential analysis yield the entire development of the boundary layer and indicate the streamwise length required for the case of the boundary layer at an infinite stagnation line to be obtained.

1. Introduction

The laminar boundary layer along the stagnation line of a swept, infinite cylinder represents a three-dimensional, similar flow in which no quantities change in the spanwise direction. For the case of an incompressible fluid the chordwise flow is described by the solution for the two-dimensional stagnation line, i.e. by the Falkner–Skan equation with $\beta = 1$, while the spanwise flow may be obtained as a second step in the calculation. Cooke (1950) has provided solutions for this case. With compressible fluids the chordwise and spanwise flows are coupled through the pressure gradient term in the chordwise momentum equation. These cases have been studied by a variety of workers; the definitive references are Reshotko & Beckwith (1957) for zero mass transfer and Beckwith (1958) for transpiration cooling.

We are concerned here with the behaviour of the boundary layer along the stagnation line of semi-infinite extent, i.e. along a cylinder with a starting edge. We shall idealize the external flow by assuming that the spanwise velocity and the chordwise velocity gradient are independent of the spanwise co-ordinate. This idealization can be realized physically in several ways; suppose on a yawed cylinder of uniform cross-section, the boundary layer is removed at some spanwise station by a suction patch or slot. Then a new boundary layer will be initiated and will develop so as to approach that relevant to the infinite cylinder far

removed from the starting station. A second way is to consider a flat plate aligned with an undisturbed flow and to imagine a uniform outflow from a plane of symmetry imposed by an external body. Again far downstream from the leading edge, we expect an asymptotic behaviour independent of streamwise distance. Clearly these cases can involve mass transfer and we shall include its effect in our analysis.

In our discussion below we shall consider explicitly the case of the cylinder, but it should be understood that the analysis applies to other flow configurations as well. We restrict ourselves to outflow from the stagnation line such as occurs on convex cylinders, i.e. we do not consider the case of inflow as might occur on concave cylinders and on flat surfaces with an appropriate, externally imposed pressure field.

Although our external flow is idealized so that comparison with experiment may be difficult, our study provides an indication of the spanwise length required for the boundary layer to approach that applicable to the infinite case. This may be of interest in a variety of applied problems.

Concerning previous work relevant to our study, references to the case of the infinite cylinder have been cited above. Moore (1957) has considered the case of flows described by the Falkner–Skan equation in the spanwise direction with small outflow in order to show that outflow delays separation. Our study is more restricted in the sense that only the case of a uniform spanwise flow is considered, but more general in that compressibility and mass transfer are included, and in that the entire development from the leading edge to far downstream in the spanwise direction is considered. Banks (1967) has recently analyzed an incompressible fluid with a linearly retarded flow in the streamwise direction in order to study the development of the stagnation point of the saddle type. Finally, Liu (1959) has considered the limiting case of a highly swept stagnation line of semi-infinite extent without mass transfer. In this case a small cross-flow approximation applies and simple similarity solutions pertain. Liu emphasizes the heat transfer aspects of the flow based on generally inaccessible solutions for the velocity field so that direct comparison with our work is not possible.†

2. Analysis

The geometry and schematic representation of the flow under consideration are shown in figure 1. We take the (x, z) plane to be the plane of symmetry, the velocity components in the external flow to be independent of x and to be $u_e, v_e y$ in the x, y direction, respectively. The quantities u_e and v_e depend on the velocity far from the cylinder, the sweep angle and the nose radius of the cylinder. If the Prandtl number is assumed to be unity, then the equations describing the boundary layer are

$$\rho \left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = \frac{\partial}{\partial z} \left(\mu \frac{\partial u}{\partial z} \right), \quad (2.1)$$

$$\rho \left(u \frac{\partial v}{\partial x} + v^2 + w \frac{\partial v}{\partial z} \right) = \rho_e v_e^2 + \frac{\partial}{\partial z} \left(\mu \frac{\partial v}{\partial z} \right), \quad (2.2)$$

† The authors are indebted to a reviewer for calling this reference to their attention.

$$\rho \left(u \frac{\partial h_s}{\partial x} + w \frac{\partial h_s}{\partial z} \right) = \frac{\partial}{\partial z} \left(\mu \frac{\partial h_s}{\partial z} \right), \quad (2.3)$$

$$\frac{\partial}{\partial x} (\rho u) + (\rho v) + \frac{\partial}{\partial z} (\rho w) = 0, \quad (2.4)$$

where the usual notation applies except for v which denotes the y derivative of the y velocity component.†

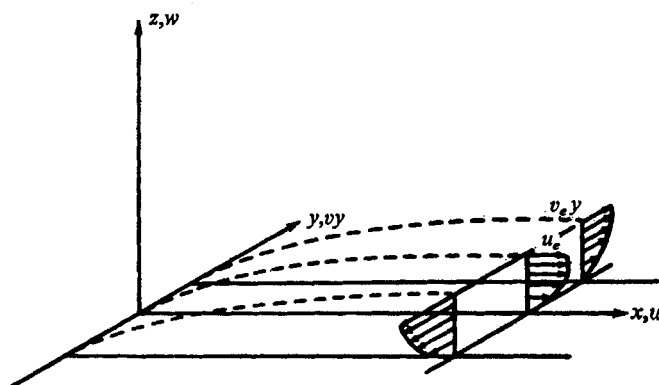


FIGURE 1. Schematic representation of the flow.

2.1. Transformed equations

If we introduce two stream functions ψ and $\hat{\Phi}$ such that

$$\rho u = \partial \psi / \partial z, \quad \rho v = \partial \hat{\Phi} / \partial z, \quad (2.5)$$

then without loss in generality, from (2.4),

$$\rho w = -(\partial \psi / \partial x) - \hat{\Phi}. \quad (2.6)$$

We also introduce new independent variables

$$\zeta \equiv (\rho_e u_e / 2\mu_e x)^{\frac{1}{2}} \int_0^z (\rho / \rho_e) dz' = \zeta(x, z),$$

$$\xi^2 = (2v_e x / u_e) \propto x,$$

and new dependent variables

$$\left. \begin{aligned} \psi &= (2\rho_e \mu_e u_e x)^{\frac{1}{2}} f(\xi, \zeta), \\ \hat{\Phi} &= v_e (2\rho_e \mu_e x / u_e)^{\frac{1}{2}} \phi(\xi, \zeta). \end{aligned} \right\} \quad (2.7)$$

Then

$$u = u_e (\partial f / \partial \zeta), \quad v = v_e (\partial \phi / \partial \zeta). \quad (2.8)$$

† It will be recognized that (2.1)–(2.4) apply only in the region of the stagnation line, i.e. that they correspond to the first, non-vanishing terms in power series in y reflecting the appropriate odd or even behaviour of the flow variables as $y \rightarrow 0$. With respect to v_e we note that Reshotko & Beckwith (1957) provide relations for estimating v_e (du_e/dx in their notation) in terms of the Mach number normal to a cylinder of nose diameter D .

If we are to include mass transfer, we shall be interested in $(\rho w)_w$; the practically interesting case in the present problem is $(\rho w)_w = \text{constant}$. With this assumption (2.6) and (2.7) yield in general

$$\kappa \xi = -f_w - \xi^{-1} \int_0^\xi \xi'^2 \check{\phi}_w d\xi', \tag{2.9}$$

where $\kappa \equiv (\rho w)_w / 2(\rho_e \mu_e v_e)^{1/2}$ is a mass transfer parameter which will be important to our study. As is usual in three-dimensional problems we may satisfy (2.9) in a variety of ways. Without loss of generality we shall set $\check{\phi}_w \equiv 0$ and take $-f_w = \kappa \xi$. The solutions so obtained may be used to construct other solutions by adding $\alpha \kappa \xi$ and $(-2\kappa \beta / \xi)$ to our f and $\check{\phi}$, respectively, where α and β are arbitrary constants subject only to the requirement that $\alpha + \beta = 0$.

The equations for $f(\xi, \zeta)$, $\check{\phi}(\xi, \zeta)$ are found to be

$$\frac{\partial^3 f}{\partial \xi^3} + (f + \xi^2 \check{\phi}) \frac{\partial^2 f}{\partial \xi^2} - \xi \left(\frac{\partial f}{\partial \zeta} \frac{\partial^2 f}{\partial \zeta \partial \xi} - \frac{\partial f}{\partial \xi} \frac{\partial^2 f}{\partial \zeta^2} \right) = 0, \tag{2.10}$$

$$\frac{\partial^2 \check{\phi}}{\partial \xi^2} + (f + \xi^2 \check{\phi}) \frac{\partial^2 \check{\phi}}{\partial \xi^2} + \xi^2 \left[\frac{\rho_e}{\rho} - \left(\frac{\partial \check{\phi}}{\partial \xi} \right)^2 \right] - \xi \left(\frac{\partial f}{\partial \zeta} \frac{\partial^2 \check{\phi}}{\partial \zeta \partial \xi} - \frac{\partial f}{\partial \xi} \frac{\partial^2 \check{\phi}}{\partial \zeta^2} \right) = 0, \tag{2.11}$$

provided, as is frequently done, we assume $\rho \mu \simeq \rho_e \mu_e$.

We take the frequently employed, approximate equation of state $\rho \sim h^{-1}$, where h is the static enthalpy; assume a constant wall enthalpy; and from a comparison of (2.1) and (2.3) deduce that a Crocco relation applies. Thus

$$(\rho_e / \rho) = [g_w + (1 - g_w)(\partial f / \partial \zeta) - \tilde{m}(\partial f / \partial \zeta)^2](1 - \tilde{m})^{-1}, \tag{2.12}$$

where $g_w \equiv (h_w / h_{s,e})$ and $\tilde{m} \equiv u_e^2 / 2h_{s,e}$, and where the subscript s, e refers to stagnation conditions external to the boundary layer.

The boundary conditions applicable to (2.10) and (2.11) are

$$\left. \begin{aligned} f(\xi, 0) = -\kappa \xi, \quad \partial f / \partial \zeta(\xi, 0) = \check{\phi}(\xi, 0) = \partial \check{\phi} / \partial \zeta(\xi, 0) = 0, \\ \partial f / \partial \zeta(\xi, \infty) = \partial \check{\phi} / \partial \zeta(\xi, \infty) = 1. \end{aligned} \right\} \tag{2.13}$$

We shall assume that no arbitrary initial data are to be prescribed at $\xi > 0$, but rather that the behaviour of (2.10) and (2.11) as $\xi \rightarrow 0$ implies acceptable initial data. Let $f_i(\zeta)$ and $\check{\phi}_i(\zeta)$ denote the solutions as $\xi \rightarrow 0$; we find

$$\left. \begin{aligned} f_i''' + f_i f_i'' = 0, \quad f_i(0) = f_i'(0) = 0, \quad f_i'(\infty) = 1, \\ \check{\phi}_i''' + f_i \check{\phi}_i'' = 0, \quad \check{\phi}_i(0) = \check{\phi}_i'(0) = 0, \quad \check{\phi}_i'(\infty) = 1, \end{aligned} \right\} \tag{2.14}$$

where ()' indicates differentiation with respect to ζ . It is clear from (2.14) that the initial profiles in both the streamwise and cross-flow directions are given by the Blasius solution, independent of the parameters κ, g_w and \tilde{m} . The physical explanation of this result appears to reside in the dominance of the shear stress over the effects of the cross-flow pressure gradient and mass transfer at the leading edge. It is from this flow that the boundary layer develops with spanwise distance, i.e. with increasing ξ , until that corresponding to the infinite swept stagnation line prevails. This same leading edge behaviour for incompressible flows is given by Moore (1957), Sowerby (1965) and Banks (1967).

From the solutions to (2.10) and (2.11), the quantities which may be used to compute the skin-friction, the heat transfer and the surface flow divergence, are $(\partial^2 f / \partial \xi^2)(\xi, 0)$ and $(\partial^2 \phi / \partial \xi^2)(\xi, 0)$, and therefore the distribution of these quantities with ξ for sets of the parameters κ , g_w and \tilde{m} will be of interest. Note that in all cases $(\partial^2 f / \partial \xi^2)(0, 0) = (\partial^2 \phi / \partial \xi^2)(0, 0) = 0.469600$.

It is of interest to observe that the independent variable ξ is analogous to ζ of Moore (1957) for his $m = 0$, i.e. for a uniform spanwise flow and that it contains the effects of cross-flow gradient in v_e , of spanwise length in x , and of spanwise flow in u_e . Thus, if we find the effect of the leading edge vanishes effectively at a given value of ξ , various combinations of v_e , x and u_e and thus of the cylindrical radius and sweep angle may be considered to achieve that particular value of ξ .

Because 'independence principles' play a useful role in three-dimensional boundary layer theory, we note (2.10) and (2.11) show that near the starting edge, i.e. for $\xi^2 \ll 1$ the spanwise boundary layer is independent of the chordwise boundary layer, but the converse is not true even for the constant density case, $g_w = 1$, $\tilde{m} = 0$. This may be contrasted with the case of a stagnation line of infinite extent (cf. § 1).

2.2. Asymptotic solution

Suppose we are sufficiently remote from the starting edge so that we may assume that no further changes with x occur. Then (2.1)–(2.4) specialize to

$$\rho w \frac{\partial u}{\partial z} = \frac{\partial}{\partial z} \left(\mu \frac{\partial u}{\partial z} \right), \quad (2.15)$$

$$\rho v^2 + \rho w \frac{\partial v}{\partial z} = \rho_e v_e^2 + \frac{\partial}{\partial z} \left(\mu \frac{\partial v}{\partial z} \right), \quad (2.16)$$

$$\rho w \frac{\partial h_s}{\partial z} = \frac{\partial}{\partial z} \left(\mu \frac{\partial h_s}{\partial z} \right), \quad (2.17)$$

$$\rho v + \frac{\partial}{\partial z} (\rho w) = 0. \quad (2.18)$$

We now introduce a new independent variable,

$$\bar{\zeta} = \left(\frac{\rho_e v_e}{\mu_e} \right)^{\frac{1}{2}} \int_0^z \frac{\rho}{\rho_e} dz = \bar{\zeta}(z),$$

and let

$$u = u_e F'_0, \quad v = v_e \Phi'_0,$$

where (') denotes in this section differentiation with respect to $\bar{\zeta}$. Then (2.18) implies

$$-\rho w = (\rho_e \mu_e v_e)^{\frac{1}{2}} \Phi_0 + \text{constant}.$$

We may determine the constant so that

$$-\Phi_0(0) = (\rho w)_w (\rho_e \mu_e v_e)^{-\frac{1}{2}} = 2\kappa. \quad (2.19)$$

Introducing $F_0(\bar{\zeta})$, $\Phi_0(\bar{\zeta})$ into (2.15) and (2.16), we find

$$F_0''' + \Phi_0 F_0'' = 0, \quad (2.20)$$

$$\Phi_0'' + \Phi_0 \Phi_0'' + \left(\frac{\rho_e}{\rho} - \Phi_0'^2 \right) = 0, \quad (2.21)$$

which are subject to the boundary conditions

$$F_0(0) = F'_0(0) = \Phi'_0(0) = 0, \quad F'_0(\infty) = \Phi'_0(\infty) = 1, \quad \Phi_0(0) = -2\kappa.$$

From a comparison of (2.15) and (2.17) we see that a Crocco relation prevails, so that, with the same approximate equation of state as assumed above, (2.12) applies with $(\partial f/\partial \zeta)$ replaced by F'_0 .

Equations (2.20) and (2.21) are those solved by Reshotko & Beckwith (1957) for $\kappa = 0$, i.e. for no mass transfer, and by Beckwith (1958) for $\kappa > 0$, i.e. for injection. Since we are interested in the mass transfer case, we shall be explicit in identifying our equations with the latter reference. For this purpose we must let $F'_0 \rightarrow \theta$, $\Phi_0 \rightarrow f$, $\tilde{m} \rightarrow 1 - t = 1 - \lambda^{-1}$ and $g_w \rightarrow t_w$ and set the β of Beckwith equal to unity.

Now from the solutions of (2.20) and (2.21) we may determine the important wall values $F''_{0,w}$, $\Phi''_{0,w}$. From the relation between ζ and ξ , $\xi = \xi\zeta$, we conclude that

$$\left. \begin{aligned} \lim_{\xi \rightarrow \infty} f_{\xi\zeta}(\xi, 0) &= \xi F''_{0,w}, \\ \lim_{\xi \rightarrow \infty} \phi_{\xi\zeta}(\xi, 0) &= \xi \Phi''_{0,w}. \end{aligned} \right\} \quad (2.22)$$

Thus, the behaviour as $\xi \rightarrow \infty$, of the important wall parameters is established. It is interesting to note that an asymptotic solution exists for the present flow for injection, suction and no mass transfer. This contrasts with the analysis of Libby (1969) for the flow at the windward plane of symmetry on a cone with uniform mass transfer. In that case, an asymptotic solution exists only for suction.

Equations (2.20) and (2.21) describe the flow far from the starting edge, i.e. as $\xi \rightarrow \infty$. Suppose we wish more information on how our asymptotic solution is approached. To get it, it is convenient to consider the equations describing the non-similar flow, i.e. (2.10) and (2.11), and to change variables, ξ, ζ to $\xi, \bar{\xi}$ and $f = (F/\xi) - \kappa\xi$, $\phi = (\Phi + 2\kappa)/\xi$. The new dependent variables F and Φ are found to be given by the equations

$$\frac{\partial^3 F}{\partial \bar{\xi}^3} + \Phi \frac{\partial^2 F}{\partial \bar{\xi}^2} - \frac{1}{\xi} \left(\frac{\partial F}{\partial \bar{\xi}} \frac{\partial^2 F}{\partial \bar{\xi} \partial \xi} - \frac{\partial F}{\partial \xi} \frac{\partial^2 F}{\partial \bar{\xi}^2} \right) = 0, \quad (2.23)$$

$$\frac{\partial^3 \Phi}{\partial \bar{\xi}^3} + \Phi \frac{\partial^2 \Phi}{\partial \bar{\xi}^2} + \left[\frac{\rho_e}{\rho} - \left(\frac{\partial \Phi}{\partial \bar{\xi}} \right)^2 \right] - \frac{1}{\xi} \left(\frac{\partial F}{\partial \bar{\xi}} \frac{\partial^2 \Phi}{\partial \bar{\xi} \partial \xi} - \frac{\partial F}{\partial \xi} \frac{\partial^2 \Phi}{\partial \bar{\xi}^2} \right) = 0, \quad (2.24)$$

subject to the boundary conditions

$$F(\xi, 0) = \frac{\partial F}{\partial \bar{\xi}}(\xi, 0) = \frac{\partial \Phi}{\partial \bar{\xi}}(\xi, 0) = 0, \quad \Phi(\xi, 0) = -2\kappa,$$

$$\frac{\partial F}{\partial \bar{\xi}}(\xi, \infty) = \frac{\partial \Phi}{\partial \bar{\xi}}(\xi, \infty) = 1.$$

The initial data must be specified at a finite value of ξ , say $\xi = \xi_i > 0$, in terms of distributions of F and Φ .

Equations (2.23) and (2.24) may be considered equivalent to the original (2.10) and (2.11) and may be integrated by numerical means for increasing ξ . Clearly

as $\xi \rightarrow \infty$, (2.23) and (2.24) reduce to (2.20) and (2.21). To examine further the asymptotic behaviour we assume that the solutions for $f(\xi, \zeta)$, $\phi(\xi, \zeta)$ have been determined to some value of ξ such that the corresponding solutions for $F(\xi, \bar{\zeta})$, $\Phi(\xi, \bar{\zeta})$ are close to the relevant asymptotic values $F_0(\bar{\zeta})$, $\Phi_0(\bar{\zeta})$ defined by (2.20) and (2.21). In this situation it is necessary to examine what information determines the approach to the asymptotic solution. It is readily found that only initial data at $\xi = \xi_i$ does so, there being no forcing influence from the boundary conditions. This may be contrasted with the case of the parabolic slab, treated by van Dyke (1964) and of the blunt wedge, treated by Chen *et al.* (1969), wherein the external flow as well as initial data determine the approach to the asymptotic solution.

Thus for the present problem we would not expect a series in inverse half or whole powers of ξ to yield any information, and indeed the assumption of a solution in the form of such a series fails. However, we would expect an eigenvalue problem, which would introduce arbitrary constants to be determined by initial data at $\xi = \xi_i$. Accordingly, we assume that the corrections to the asymptotic values are given by a succession of pairs of functions $F_1(\xi, \bar{\zeta})$, $\Phi_1(\xi, \bar{\zeta})$, $F_2(\xi, \bar{\zeta})$, $\Phi_2(\xi, \bar{\zeta})$, etc., where the first pair are defined by

$$\frac{\partial^3 F_1}{\partial \bar{\zeta}^3} + \Phi_0 \frac{\partial^2 F_1}{\partial \bar{\zeta}^2} + F_0'' \Phi_1 - \frac{1}{\xi} \left(F_0' \frac{\partial^2 F_1}{\partial \bar{\zeta} \partial \xi} - F_0'' \frac{\partial F_1}{\partial \bar{\zeta}} \right) = 0, \tag{2.25}$$

$$\begin{aligned} \frac{\partial^3 \Phi_1}{\partial \bar{\zeta}^3} + \Phi_0 \frac{\partial^2 \Phi_1}{\partial \bar{\zeta}^2} + \Phi_0'' \Phi_1 + \left[(1 - g_w)(1 - \tilde{m})^{-1} \frac{\partial \Phi_1}{\partial \bar{\zeta}} \right. \\ \left. - 2\tilde{m}(1 - \tilde{m})^{-1} F_0' \frac{\partial F_1}{\partial \bar{\zeta}} - 2\Phi_0' \frac{\partial \Phi_1}{\partial \bar{\zeta}} \right] \\ - \frac{1}{\xi} \left(F_0' \frac{\partial^2 \Phi_1}{\partial \bar{\zeta} \partial \xi} - \Phi_0'' \frac{\partial \Phi_1}{\partial \bar{\zeta}} \right) = 0, \tag{2.26} \end{aligned}$$

where the boundary conditions are homogeneous and where initial data on F_1 and Φ_1 are specified at $\xi = \xi_i$.

Separation of variables applied to (2.25) and (2.26) leads to a ξ -wise variation of the form $\exp(-\frac{1}{2}\lambda\xi^2)$ and to a $\bar{\zeta}$ variation resulting in a higher-order eigenvalue problem for which the usual techniques permitting proof of the realness of the eigenvalues to be established do not appear to apply. Thus λ must in general be considered complex although physical arguments could be invoked to justify the assumption of λ real. Accordingly, a major effort (one beyond the scope of the present paper) is required to obtain more information concerning the approach to the asymptotic solution.

2.3. Method of solutions: starting series

We return now to our original problem. Either (2.10) and (2.11) or (2.23) and (2.24) with (2.12), and with the initial and boundary conditions, may be solved by numerical methods. To establish the behaviour near the starting station we develop first a series solution in ξ . The absence of information concerning the approach to the solution for $\xi \rightarrow \infty$ precludes using a few terms in this series to determine the boundary-layer behaviour over the entire range of ξ . Accordingly,

as a second approach we apply straightforward numerical analysis to (2.23) and (2.24) with $\bar{\xi} \equiv \frac{1}{2}\xi^2$ and ζ as independent variables and with initial data supplied by the series solution at some small positive value of $\bar{\xi}$.

For the series solution we take

$$f(\xi, \zeta) = \sum_{n=0}^{\infty} \xi^n f_n(\zeta), \quad \check{\phi}(\xi, \zeta) = \sum_{n=0}^{\infty} \xi^n \check{\phi}_n(\zeta), \tag{2.27}$$

where $f_0 \equiv \phi_0$ is the Blasius solution. Substitution into (2.10) and (2.11) and collection of powers of ξ results in a hierarchy of equations of the form,

$$\left. \begin{aligned} L_n f_n &\equiv f_n''' + f_0 f_n'' - n f_0' f_n' + (n+1) f_0'' f_n = H_n \quad (n \geq 1), \\ \mathcal{L}_n \check{\phi}_n &\equiv \check{\phi}_n''' + f_0 \check{\phi}_n'' - n f_0' \check{\phi}_n' = \mathcal{H}_n \quad (n \geq 1), \end{aligned} \right\} \tag{2.28}$$

where again (') denotes differentiation with respect to ζ , and where H_n, \mathcal{H}_n are given functions of previously determined terms in the series. In particular, H_n depends on $f_0, f_1, \dots, f_{n-1}, \check{\phi}_1, \check{\phi}_2, \dots, \check{\phi}_{n-2}$ and their derivatives, while \mathcal{H}_n depends on $f_0, f_1, \dots, f_n, \check{\phi}_1, \check{\phi}_2, \dots, \check{\phi}_{n-1}$. To give some indication of their form we write the first several:

$$\begin{aligned} H_n &= 0 \quad (n = 1), \\ &= -2f_1 f_1'' + f_1'^2 - f_0 f_0'' \quad (n = 2), \\ &= -2f_1 f_2'' - 3f_1' f_2' - f_0 f_1'' - f_0'' \check{\phi}_1 + 3f_1' f_2' \quad (n = 3), \\ \mathcal{H}_n &= -2f_0'' f_1 \quad (n = 1), \\ &= -3f_0'' f_2 - 2f_1 \check{\phi}_1'' - f_0 f_0'' + f_1' \check{\phi}_1' - g_w(1 - \tilde{m})^{-1}(1 - f_0') \\ &\quad - (1 - \tilde{m})^{-1} f_0'(1 - f_0') \quad (n = 2), \\ &= -4f_0'' f_3 - 3f_1 \check{\phi}_2'' - (3f_2 + f_0) \check{\phi}_1'' - f_0'' \check{\phi}_1 - 2f_0' f_1' \\ &\quad + 2f_0' \check{\phi}_1' + 2f_1' \check{\phi}_2' + f_2' \check{\phi}_1' + g_w(1 - \tilde{m})^{-1} f_1' \\ &\quad - (1 - \tilde{m})^{-1} f_1'(1 - 2f_0') \quad (n = 3). \end{aligned}$$

All the initial and boundary conditions are handled by f_0 except for

$$f(\xi, 0) = -\kappa \xi.$$

Thus,

$$f_1(0) = -\kappa, \quad f_n(0) = 0 \quad \text{for } n \geq 2,$$

and

$$\check{\phi}_n(0) = \check{\phi}_n'(0) = f_n'(0) = f_n'(\infty) = \check{\phi}_n'(\infty) = 0 \quad (n \geq 1).$$

Equations (2.28) may be solved singly and successively according to the sequence $f_1, \check{\phi}_1, f_2, \check{\phi}_2, \dots$. The three parameters, κ, g_w , and \tilde{m} , appear as $\kappa, g_w(1 - \tilde{m})^{-1}$ and $(1 - \tilde{m})^{-1}$. Because the operators L_n and \mathcal{L}_n are free of parameters, depending as they do only on f_0 and its derivatives, we can construct solutions on the basis of superposition of universal functions weighted by these parameters and by powers of these parameters. If we let $P_1 \equiv g_w(1 - \tilde{m})^{-1}$ and $P_2 \equiv (1 - \tilde{m})^{-1}$, then a convenient triple subscript notation leads to

$$\left. \begin{aligned} f_n(\xi) &= \sum_{j=0,1,\dots} \kappa^j [f_{n,0,j}(\xi) + P_1 f_{n,1,j}(\xi) + P_2 f_{n,2,j}(\xi)], \\ \check{\phi}_n(\xi) &= \sum_{j=0,1,\dots} \kappa^j [\check{\phi}_{n,0,j}(\xi) + P_1 \check{\phi}_{n,1,j}(\xi) + P_2 \check{\phi}_{n,2,j}(\xi)]. \end{aligned} \right\} \tag{2.29}$$

With this form of solution, it is found that

$$\phi_{1,0,1} = f_{1,0,1} + 1, \quad \check{\phi}_{n,0,j} \equiv f_{n,0,j} \quad (n \geq 2).$$

We have computed f_n and $\check{\phi}_n$ for $n = 1, 2, 3, 4$ and present in table 1 the important values $f''_{n,i,j}(0)$, $\check{\phi}''_{n,i,j}(0)$.† Values not shown for $n \leq 4$ correspond to terms in the series identically zero, e.g. $f_{1,0,0} \equiv f_{3,0,0} \equiv \check{\phi}_{1,0,0} \equiv \check{\phi}_{3,0,0} \equiv 0$. Although it is impractical and perhaps unnecessary to give graphically all the solutions, we do show in figure 2 the ‘velocity profiles’, $f'_{n,i,j}$, $\check{\phi}'_{n,i,j}$ for the special case $i = 0$. We see, then, the typical profiles obtained from series solutions for boundary-layer problems.

i	j n	0	1	2	3	4
0	1	—	-1.2243	—	—	—
		—	-1.2243	—	—	—
	2	0.1174	—	0.5510	—	—
		0.1174	—	0.5510	—	—
	3	—	-0.1089	—	0.2258	—
		—	-0.1089	—	0.2258	—
	4	0.004892	—	-0.09438	—	0.07427
		0.004892	—	-0.09438	—	0.07427
1	2	—	—	—	—	—
		0.6191	—	—	—	—
	3	—	0.2485	—	—	—
		0.04677	—	—	—	—
4	-0.1080	—	0.1433	—	—	
	—	—	—	—	—	
2	2	—	—	—	—	—
		0.1488	—	—	—	—
	3	—	—	—	—	—
		—	-0.1304	—	—	—
4	0.01729	—	—	—	—	
	-0.01991	—	-0.1654	—	—	

TABLE 1. Crucial wall values for series solution $f''_{n,i,j}(0)$, $\check{\phi}''_{n,i,j}(0)$

2.4. Numerical solution for ξ large

The few terms in the series presented above may be used to provide the initial data for a numerical treatment of (2.23) and (2.24), i.e. of the equations appropriate for large ξ . Equations (2.23) and (2.24) are put in an implicit, difference-differential form, so that solutions at $\xi = \xi_n$ are obtained from the solution of a set of ordinary differential equations which have ζ as the independent variable.

† A few of our results can be compared with Sowerby (1965), e.g. $f_{2,0,0} = f_{11}$, $\check{\phi}_{2,1,0} + \check{\phi}_{2,2,0} = g_{11}$, where the right-hand sides are his notation. The two calculations agree to all numbers available to us. In addition, the results for $f_{n,0,j}$, $n = j$ can be compared with Libby & Chen (1965), since our equations are identical to theirs if $\kappa\xi = -\xi$. The calculations agree to 3 and 4 significant figures.

Denote by ${}^{(n)}F(\bar{\xi})$, ${}^{(n)}\Phi(\bar{\xi})$ the solutions for F and Φ respectively at $\xi = \xi_n$. The ordinary differential equations for ${}^{(n)}F$ and ${}^{(n)}\Phi$ are linearized according to the approximation indicated by, e.g.

$${}^{(n)}F^2 \simeq 2{}^{(n-1)}F{}^{(n)}F - ({}^{(n-1)}F)^2. \tag{2.30}$$

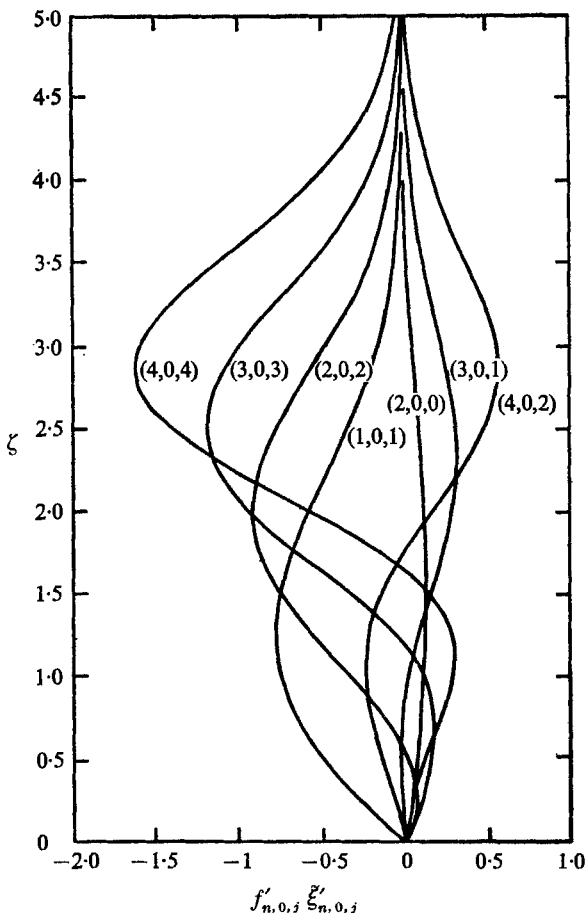


FIGURE 2. The $f'_{n,0,j}$, $\tilde{\phi}'_{n,0,j}$ velocity profiles. ———, series; — — —, difference-differential.

Thus the two-point boundary conditions for ${}^{(n)}F$, ${}^{(n)}\Phi$ may be satisfied by superposition of two complementary solutions and a particular solution, each obtained numerically. Finally, we note that we use a second-order difference formula for the $\bar{\xi}$ derivatives as indicated by

$${}^{(n)}\left(\frac{\partial F}{\partial \bar{\xi}}\right) \simeq \frac{{}^{(n-2)}F - 4{}^{(n-1)}F + 3{}^{(n)}F}{2\Delta}, \tag{2.31}$$

where Δ is the spacing in $\bar{\xi}$. This representation requires the series solution to be employed at two values of ξ .

The calculations were carried out for increasing $\bar{\xi}$ until values of the wall shear parameters $F_{\bar{\xi},w}^{\bar{\xi}}$ and $\Phi_{\bar{\xi},w}^{\bar{\xi}}$ at two successive stations were both unaltered within 1% of $\Delta F_{\bar{\xi},w}^{\bar{\xi}}$ and $\Delta \Phi_{\bar{\xi},w}^{\bar{\xi}}$, respectively.

We have compared our numerical results for $\bar{\xi}$ large with seven of the solutions of Beckwith (1958) involving combinations of κ , g_w and \bar{m} . Our 'asymptotic' values of $F_{\bar{\xi},w}^{\bar{\xi}}$ and $\Phi_{\bar{\xi},w}^{\bar{\xi}}$ agree in each case to at least two significant figures with those of Beckwith, so that we consider our numerical procedures to be valid.

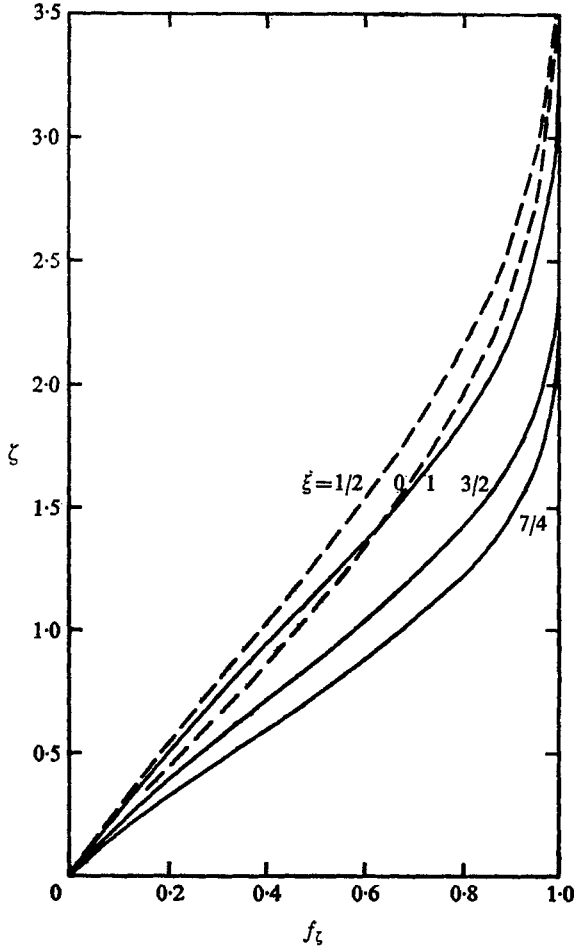


FIGURE 3. Development of the streamwise velocity profile: $\kappa = 0.25$, $g_w = 0.5$, $\bar{m} = 0.375$. — — —, series; — — —, difference-differential.

3. Results

We discuss the results of our analysis in terms of the behaviour of the boundary layer near the starting station, i.e. for $0 \leq \xi \leq \frac{1}{2}$ and then for the downstream region, $\xi > 1$.

From the entries in table 1 it is instructive to construct expressions for $f_{\zeta\zeta, w}$ and $\check{\phi}_{\zeta\zeta, w}$ near $\xi = 0$, namely,

$$\left. \begin{aligned} f_{\zeta\zeta, w} &\simeq 0.4696 - 1.2243(\kappa\xi) + (0.1174 + 0.5510\kappa^2)\xi^2 + O(\xi^3), \\ \check{\phi}_{\zeta\zeta, w} &\simeq 0.4696 - 1.2243(\kappa\xi) + (0.1174 + 0.5510\kappa^2 \\ &\quad + 0.6191P_1 + 0.1488P_2)\xi^2 + O(\xi^3). \end{aligned} \right\} \quad (3.1)$$

In these expressions, the terms involving $\kappa\xi$ describe the effect of mass transfer alone, while those free of κ account for the effect of outflow from the stagnation line. From (3.1) we see that mass transfer dominates the initial development of the wall shear from that given by the Blasius solution, and that as ξ increases the second term, which is always positive and which contains an effect of outflow alone, will dominate.

This behaviour is shown in a different form in figure 3; for $\kappa = 0.25$, $g_w = 0.5$, $\tilde{m} = 0.375$, † one of the cases solved by Beckwith (1958) for the infinite stagnation line, we show the streamwise velocity profile $f_{\zeta}(\xi, \zeta)$ for several values of ξ as given by the five available terms in the series solution for $\xi = 0, \frac{1}{2}$, and by the difference-differential solution for $\xi \simeq 1, \frac{3}{2}, \frac{7}{4}$. The boundary layer is seen to develop initially under the influence of injection somewhat faster than $x^{\frac{1}{2}}$, but as ξ increases there dominates the cross-flow effect leading to thinning of the layer in terms of ζ or to the approach to an invariance with x .

The downstream behaviour of the wall shear parameters, as given by the series solution and the difference-differential solution, are shown in figures 4 and 5 in terms of $f_{\zeta\zeta, w} - \xi F''_{0, w}$ and $\check{\phi}_{\zeta\zeta, w} - \xi \Phi''_{0, w}$ versus ξ , where for self-consistency we take for $F''_{0, w}$ and $\Phi''_{0, w}$ the values of $F''_{\xi\bar{\zeta}, w}$ and $\Phi''_{\xi\bar{\zeta}, w}$ we obtain from our numerical solution when further changes with ξ are negligible. The results shown in these figures are typical of the seven cases, which we have computed following Beckwith (1958), and which involve various combinations of injection rate, wall enthalpy and sweep. In all cases, the cross-flow shear adjusts more rapidly to its asymptotic value than does the streamwise shear. As might be expected, increase in sweep (\tilde{m} increased), increase in wall enthalpy, and decrease in injection rate reduce the length in terms of ξ required by the boundary layer to adjust to its asymptotic behaviour. However, we find for all cases we have considered, that both shear parameters are within a few per cent of their asymptotic values for ξ approximately equal to 1.5 to 2. This is an applied result of some interest. ‡

We observe from figures 4 and 5 that the difference-differential results indicate a smooth, rapid approach of the wall shear parameters to their asymptotic values suggesting that the eigenvalues associated with (2.25) and (2.26) may be real or have dominant positive real parts. If they are assumed to be only real, a numerical scheme to find them and their associated eigenfunctions can be readily established. The fact that the quantities $f_{\zeta\zeta, w} - \xi F''_{0, w}$ and $\check{\phi}_{\zeta\zeta, w} - \xi \Phi''_{0, w}$ approach zero as $\xi \rightarrow \infty$ implies that there is no zero eigenvalue and that there is no ambiguity in the origin of the ξ co-ordinate.

† We consider this to be a typical case since none of the parameters defining a flow are zero.

‡ Some of these conclusions have been obtained by considering the values of $\bar{\zeta}$ at which our difference-differential calculation satisfied our termination criterion.

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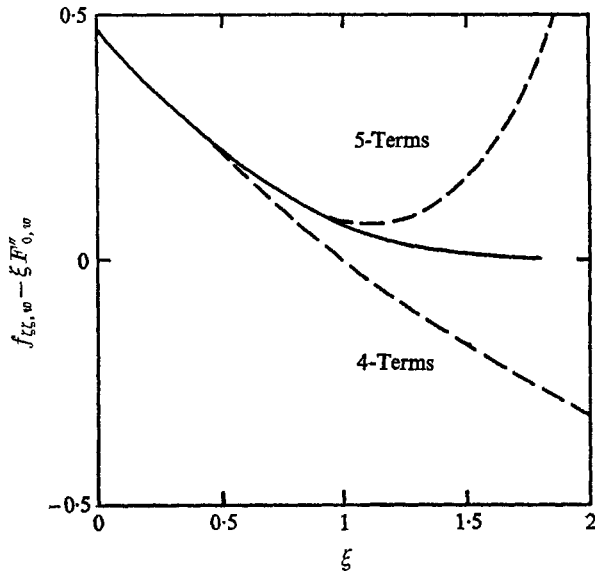


FIGURE 4. Approach of streamwise shear parameter to its asymptotic value: $\kappa = 0.25$, $g_w = 0.5$, $\tilde{m} = 0.375$. ———, series; —, difference-differential.

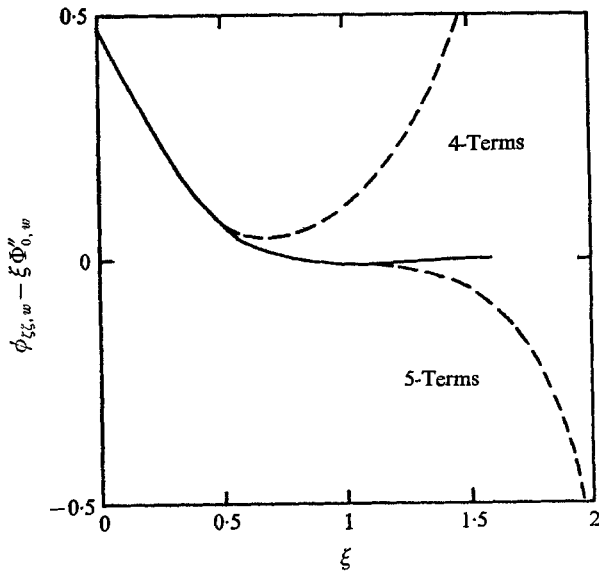


FIGURE 5. Approach of cross-flow shear parameter to its asymptotic value: $\kappa = 0.25$, $g_w = 0.5$, $\tilde{m} = 0.375$.

difference-differential solution, a discrepancy which caused us to re-examine our numerical analysis and to uncover a minor error in the ξ^4 terms.

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